Reprinted from
NON-CLASSICAL SHELL PROBLEMS
Proceedings of the I.A.S.S. Symposium
Warsaw, September 2-5, 1963

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NORTH-HOLLAND PUBLISHING COMPANY
Amsterdam
PWN—POLISH SCIENTIFIC PUBLISHERS
Warszawa
1964

VISCOELASTIC DAMPING OF VIBRATIONS OF SANDWICH PLATES AND SHELLS[1]

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1. Introduction

In this paper we shall discuss a sequence of four topics which culminate in a study of the viscoelastic damping of vibrations of sandwich plates and shells. The paper is thus divided into four parts. In the first part a new variational principle is presented, which may be considered as a generalized Hamilton's principle. The principle is made use of in the second part of the paper in the derivation of the complete system of equations for a sandwich cylindrical shell, which are reducible to those of a sandwich plate as a special case. The way the equations are derived is also new. In the third part of the paper the equations of the sandwich plate and cylindrical shell are used to investigate the undamped vibrations of these structures. Of particular interest are an analysis of the coupling between the flexural and extensional motions of the sandwich cylindrical shell and a demonstration of the importance of transverse shear deformation in the vibrations of the sandwich plate and cylindrical shell. In the last part of the paper the effectiveness of viscoelastic damping of vibrations in the sandwich plate and cylindrical shell is investigated through the use of the concept of the damping parameter.

2. Generalized Hamiltons Principle

We shall carry out the variation in the following equation:

$$\delta\int_{t_0}^{t_1} L\mathrm{d}t = \delta\int_{t_0}^{t_1}(T-U+W)\mathrm{d}t = 0 \tag{1}$$

[1] This study was supported by the United States Air Force Office of Scientific Research.

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where $L = T - U + W$ may be called the generalized Lagrangian function and

$$T = \int_V \frac{1}{2} \rho \dot{u}_i \dot{u}_i \, \mathrm{d}V,$$

$$U = \int_V (\sigma_{ij} e_{ij} - \sigma_{ij} \varepsilon_{ij} + E) \, \mathrm{d}V,$$

$$W = \int_V f_i u_i \, \mathrm{d}V + \int_{S_p} \bar{p}_i u_i \, \mathrm{d}S + \int_{S_u} p_i (u_i - \bar{u}_i) \, \mathrm{d}S.$$

Cartesian tensor notation and the summation convention for repeated indices have been adopted. In the above equations, ρ is the density, u_i the displacement vector, e_{ij} the non-linear strain tensor, σ_{ij} the Trefftz stress tensor, ε_{ij} are expressions of e_{ij} as functions of the derivatives of u_i, E is the strain energy density which is assumed to exist and is a function of e_{ij}, f_i the body force, and p_i the surface traction. In addition, t_0 and t_1 are two instants of time t, overdot indicates differentiation with respect to t, overbar the prescribed value of a quantity, V the volume of the body under consideration, S_p that part of the surface S of the body on which traction is prescribed, and S_u that part of S on which displacement is prescribed.

The variations of the displacements, strains, and stresses are taken independently. We thus have

$$\delta \int_{t_0}^{t_1} T \mathrm{d}t = \int_V \rho \dot{u}_i \delta u_i \mathrm{d}V \Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} \mathrm{d}t \int_V \rho \ddot{u}_i \delta u_i dV, \tag{2}$$

$$\delta \int_{t_0}^{t_1} U \mathrm{d}t = \int_{t_0}^{t_1} \mathrm{d}t \int_V \left[\sigma_{ij} \delta \varepsilon_{ij} + (\varepsilon_{ij} - e_{ij}) \delta \sigma_{ij} - \left(\sigma_{ij} - \frac{\partial E}{\partial e_{ij}} \right) \delta e_{ij} \right] \mathrm{d}V, \tag{3}$$

$$\delta \int_{t_0}^{t_1} W \mathrm{d}t = \int_{t_0}^{t_1} \left[\int_V f_i \delta u_i \mathrm{d}V + \int_{S_p} \bar{p}_i \delta u_i dS + \int_{S_u} (u_i - \bar{u}_i) \delta p_i \mathrm{d}S \right] \mathrm{d}t. \tag{4}$$

The variation of the displacement δu_i will be assumed to vanish at $t = t_0$ and $t = t_1$ as is usual in Hamilton's principle; the first term on the right-hand side of eq. (2) is therefore zero. Since ε_{ij} are expressions involving the derivatives of displacements, they can always be represented in terms of the linear strains e_{mn} and rotations ω_{mn}:

$$\varepsilon_{ij} = \varepsilon_{ij}(e_{mn}, \omega_{mn}) \tag{5}$$

with

$$e_{mn} = \frac{1}{2}(u_{m,n} + u_{n,m})$$

$$\omega_{mn} = \frac{1}{2}(u_{m,n} - u_{n,m}) \tag{6}$$

where subscripts following a comma indicate differentiation with respect to the corresponding coordinates. Introducing e_{mn} and ω_{mn} and making use of Gauss' theorem, we find

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V = \frac{1}{2}\int_{S_p}\left[\sigma_{ij}\left(\frac{\partial\varepsilon_{ij}}{\partial e_{mn}} + \frac{\partial\varepsilon_{ij}}{\partial\omega_{mn}}\right)\nu_n\delta_{lm}\right.$$
$$\left. + \sigma_{ij}\left(\frac{\partial\varepsilon_{ij}}{\partial e_{mn}} - \frac{\partial\varepsilon_{ij}}{\partial\omega_{mn}}\right)\nu_m\delta_{ln}\right]\delta u_{el}\mathrm{d}S - \frac{1}{2}\int_V\left[\left[\left\{\sigma_{ij}\left(\frac{\partial\varepsilon_{ij}}{\partial e_{mn}} + \frac{\partial\varepsilon_{ij}}{\partial\omega_{mn}}\right)\right]\right]_{,n}\delta_{lm}\right.$$
$$\left. + \left\{\sigma_{ij}\left(\frac{\partial\varepsilon_{ij}}{\partial e_{mn}} - \frac{\partial\varepsilon_{ij}}{\partial\omega_{mn}}\right)\right\}_{,m}\delta_{ln}\right]\delta u_l\mathrm{d}V, \tag{7}$$

the left-hand side of which has appeared in eq. (3) and in which δ_{lm} is the Kronecker delta and $\nu_n = \cos(\nu, n)$. By virtue of eqs. (2), (3), (4), and (7), eq. (1) becomes

$$\int_{t_0}^{t_1}\mathrm{d}t\int_V\left[\frac{1}{2}\left\{\sigma_{ij}\left(\frac{\partial\varepsilon_{ij}}{\partial e_{mn}} + \frac{\partial\varepsilon_{ij}}{\partial\omega_{mn}}\right)\right\}_{,n}\delta_{lm}\right.$$
$$\left. + \frac{1}{2}\left\{\sigma_{ij}\left(\frac{\partial\varepsilon_{ij}}{\partial e_{mn}} - \frac{\partial\varepsilon_{ij}}{\partial\omega_{mn}}\right)\right\}_{,m}\delta_{ln} + f_l - \varrho\ddot{u}_l\right]\delta u_l\,\mathrm{d}V$$
$$- \int_{t_0}^{t_1}\mathrm{d}t\int_{S_p}\left[\frac{1}{2}\sigma_{ij}\left(\frac{\partial\varepsilon_{ij}}{\partial e_{mn}} + \frac{\partial\varepsilon_{ij}}{\partial\omega_{mn}}\right)\nu_n\delta_{lm}\right.$$
$$\left. + \frac{1}{2}\sigma_{ij}\left(\frac{\partial\varepsilon_{ij}}{\partial e_{mn}} - \frac{\partial\varepsilon_{ij}}{\partial\omega_{mn}}\right)\nu_m\delta_{ln} - \overline{p}_l\right]\delta u_l\mathrm{d}S$$
$$+ \int_{t_0}^{t_1}\mathrm{d}t\int_V\left(\sigma_{ij} - \frac{\partial E}{\partial\epsilon_{ij}}\right)\delta\epsilon_{ij}\mathrm{d}V$$
$$+ \int_{t_0}^{t_1}\mathrm{d}t\int(\epsilon_{ij} - \varepsilon_{ij})\delta\sigma_{ij}\mathrm{d}V + \int_{t_0}^{t_1}\mathrm{d}t\int_{S_u}(u_l - \overline{u}_l)\delta p_l\mathrm{d}S = 0. \tag{8}$$

Since the variations δu_l, $\delta\epsilon_{ij}$, $\delta\sigma_{ij}$ are arbitrary throughout the volume V of the body, δu_l arbitrary on S_p, and δp_l arbitrary on S_u, their coefficients in the five integrands in eq. (8) must vanish independently, which

yield in succession the stress equations of motion, stress boundary conditions, stress-strain relations, strain-displacement relations, and displacement boundary conditions. Depending on the form of ε_{ij}, these may be seen to represent the complete system of equations for infinitesimal or finite elastic deformations. We have thus proved the generalized Hamilton's principle, which may be stated as follows:

"The displacements, strains, and stresses (defined in the manner of Trefftz) over the time interval from t_0 to t_1 which satisfy the equations of motion and the stress-strain-displacement relations throughout the volume V of the body and the boundary conditions of prescribed tractions over S_p and prescribed displacements over S_u are determined by the vanishing of the variation of the time integral of the generalized Lagrangian function over the time interval; provided that the variations of the displacements, strains, and stresses be taken independently and simultaneously, that the variations of the displacements vanish at t_0 and t_1 throughout the body, and that the variations of the displacements and stresses be in consistence with the prescribed boundary conditions".

Equations (1) and (8) will be referred to as the generalized Hamilton's principle and generalized variational equation of motion, respectively. If the variations are restricted to those of displacements, eq. (8) is seen to reduce to the ordinary variational equation of motion.

3. Equations of Sandwich Plate and Cylindrical Shell

The generalized Hamilton's principle and the associated variational equation of motion are most general in that they are applicable to finite as well as infinitesimal deformations. In the following we shall make use of the generalized variational equation of motion in the derivation of linear equations of a thin sandwich cylindrical shell. The cylindrical coordinates x, $s = a\theta$ and $r = z + a$ are employed. They are in the longitudinal, circumferential, and radial directions, respectively, of the shell whose middle surface has the radius a. In the radial or thickness direction the inner face, core, and outer face layers of the shell extend from $z = -h$ to $z = -h_1$, $z = -h_1$ to $z = h_1$, and $z = h_1$ to $z = h$, respectively. The core thus has a thickness $2h_1$, each of the two face layers a thickness $h_2 = h - h_1$, and the total thickness of the shell is $2h$. The shell is closed in the s-direction and has a length l in the x-direction.

For the sandwich cylindrical shell we rewrite eq. (8) in its linearized version in cylindrical coordinates as follows:

$$
\begin{aligned}
&\int_{t_0}^{t_1} \mathrm{d}t \sum_{i=1}^{3} \iiint \Big\{\Big[\frac{\partial \sigma_{xxi}}{\partial x} + \frac{a}{r}\frac{\partial \sigma_{sxi}}{\partial s} + \frac{\partial \sigma_{zxi}}{\delta z} + \frac{\partial \sigma_{zxi}}{r} - \varrho_i \ddot{u}_{xi}\Big]\delta u_{xi} \\
&\quad + \Big[\frac{a}{r}\frac{\partial \sigma_{ssi}}{\partial s} + \frac{\partial \sigma_{xsi}}{\partial x} + \frac{\partial \sigma_{zsi}}{\partial z} + \frac{2\sigma_{zsi}}{r} - \varrho_i \ddot{u}_{si}\Big]\delta u_{si} \\
&\quad + \Big[\frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zzi} - \sigma_{ssi}}{r} + \frac{\delta \sigma_{xzi}}{\partial x} + \frac{a}{r}\frac{\partial \sigma_{szi}}{\partial s} - \varrho_i \ddot{u}_{zi}\Big]\delta u_{zi}\Big\} \\
&\quad \times \mathrm{d}x\,\mathrm{d}s\left(1 + \frac{z}{a}\right)\mathrm{d}z \\
&- \int_{t_0}^{t_1} \mathrm{d}t \sum_{i=1}^{3} \iint \{[\sigma_{xxi}\nu_x + \sigma_{sxi}\nu_s + \sigma_{zxi}\nu_z - \bar{p}_{xi}]\delta u_{xi} \\
&\quad + [\sigma_{xsi}\nu_x + \sigma_{ssi}\nu_s + \sigma_{zsi}\nu_z - \bar{p}_{si}]\delta u_{si} \\
&\quad + [\sigma_{xzi}\nu_x + \sigma_{szi}\nu_s + \sigma_{zzi}\nu_z - \bar{p}_{zi}]\delta u_{zi}\}\,\mathrm{d}S_i \\
&+ \int_{t_0}^{t_1} \mathrm{d}t \sum_{i=1}^{3} \iiint \{[\sigma_{xxi} - (\lambda_i + 2\mu_i)\epsilon_{xxi} - \lambda_i(\epsilon_{ssi} + \epsilon_{zzi})]\delta\epsilon_{xxi} \\
&\quad + [\sigma_{ssi} - (\lambda_i + 2\mu_i)\epsilon_{ssi} - \lambda_i(\epsilon_{xxi} + \epsilon_{zzi})]\delta\epsilon_{ssi} \\
&\quad + [\sigma_{zzi} - (\lambda_i + 2\mu_i)\epsilon_{zzi} - \lambda_i(\epsilon_{xxi} + \epsilon_{ssi})]\delta\epsilon_{zzi} \\
&\quad + [\sigma_{xsi} - \mu_i\epsilon_{xsi}]\delta\epsilon_{xsi} + [\sigma_{xzi} - \mu_i\epsilon_{xzi}]\delta\epsilon_{xzi} \\
&\quad + [\sigma_{szi} - \mu_i\epsilon_{szi}]\delta\epsilon_{szi}\}\,\mathrm{d}x\,\mathrm{d}s\left(1 + \frac{z}{a}\right)\mathrm{d}z \\
&+ \int_{t_0}^{t_1} \mathrm{d}t \sum_{i=1}^{3} \iiint \Big\{\Big[\epsilon_{xxi} - \frac{\partial u_i}{\partial x}\Big]\delta\sigma_{xxi} + \Big[\epsilon_{ssi} - \frac{a}{r}\Big(\frac{\partial u_{si}}{\partial s} + \frac{u_{zi}}{a}\Big)\Big]\delta\sigma_{ssi} \\
&\quad + \Big[\epsilon_{zzi} - \frac{\partial u_{zi}}{\partial z}\Big]\delta\sigma_{zzi} + \Big[\epsilon_{xsi} - \Big(\frac{\partial u_{si}}{\partial x} + \frac{a}{r}\frac{\partial u_{xi}}{\partial s}\Big)\Big]\delta\sigma_{xsi} \\
&\quad + \Big[\epsilon_{xzi} - \Big(\frac{\partial u_{zi}}{\partial x} + \frac{\partial u_{xi}}{\partial z}\Big)\Big]\delta\sigma_{xzi} \\
&\quad + \Big[\epsilon_{szi} - \Big(\frac{\partial u_{si}}{\partial z} + \frac{a}{r}\frac{\partial u_{zi}}{\partial s} - \frac{u_{si}}{r}\Big)\Big]\delta\sigma_{szi}\Big\}\,\mathrm{d}x\,\mathrm{d}s\left(1 + \frac{z}{a}\right)\mathrm{d}z \\
&+ \int_{t_0}^{t_1} \mathrm{d}t \sum_{i=1}^{3} \iint_{S_u} \{[u_{xi} - \bar{u}_{xi}]\,\delta(\sigma_{xxi}\nu_x + \sigma_{sxi}\nu_s + \sigma_{zxi}\nu_z) \\
&\quad + [u_{zi} - \bar{u}_{si}]\,\delta(\sigma_{xsi}\nu_x + \sigma_{ssi}\nu_s + \sigma_{zsi}\nu_z) \\
&\quad + [u_{zi} - \bar{u}_{zi}]\,\delta(\sigma_{xzi}\nu_x + \sigma_{szi}\nu_s + \sigma_{zzi}\nu_z)\}\,\mathrm{d}s_i = 0 \qquad (9)
\end{aligned}
$$

where the subscript $i = 1$, 2, or 3 refers to the core, inner or outer face layer of the sandwich, overbars denote prescribed quantities, and S_p and

S_u denote those portions of the surface on which traction and displacement, respectively, are prescribed. Other notations in eq. (9) are the usual ones.

The displacements are assumed in the form

$$
\begin{aligned}
&u_{x1} = u + z\psi, \qquad u_{x2}, u_{x3} = u \mp h_1 \psi,\\
&u_{s1} = v + z\varphi, \qquad u_{s2}, u_{s3} = v \mp h_1 \varphi,\\
&u_{z1} = u_{z2} = u_{z3} = w
\end{aligned}
\tag{10}
$$

or

$$
\begin{aligned}
&u_{x1} = u_{x1}^{(0)} + z u_{x1}^{(1)}, \qquad u_{x2} = u_{x2}^{(0)}, \qquad u_{x3} = u_{x3}^{(0)},\\
&u_{s1} = u_{s1}^{(0)} + z u_{s1}^{(1)}, \qquad u_{s2} = u_{s2}^{(0)}, \qquad u_{s3} = u_{s3}^{(0)},\\
&u_{z1} = u_{z1}^{(0)}, \qquad u_{z2} = u_{z2}^{(0)}, \qquad u_{z3} = u_{z3}^{(0)}.
\end{aligned}
\tag{11}
$$

The face layers have thus been taken to be membranes. In a consistent manner the strains are taken as

$$
\begin{aligned}
&\epsilon_{xx1} = e_{xx1}^{(0)} + z e_{xx1}^{(1)}, \qquad \epsilon_{xx2} = e_{xx2}^{(0)}, \qquad \epsilon_{xx3} = e_{xx3}^{(0)},\\
&\epsilon_{ss1} = \frac{a}{r}\, e_{ss1}^{(0)} + z e_{ss1}^{(1)}, \qquad \epsilon_{ss2} = \frac{a}{r} e_{ss2}^{(0)}, \qquad \epsilon_{ss3} = \frac{a}{r} e_{ss3}^{(0)},\\
&\epsilon_{zz1} = e_{zz1}^{(0)} + z e_{zz1}^{(1)}, \qquad \epsilon_{zz2} = e_{zz2}^{(0)}, \qquad \epsilon_{zz3} = e_{zz3}^{(0)},\\
&\epsilon_{xs1} = e_{xs1}^{(0)} + z e_{xs1}^{(1)} + \frac{a}{r}\left(e_{sx1}^{(0)} + e_{sx1}^{(1)}\right), \qquad \epsilon_{xs2} = e_{xs2}^{(1)} + \frac{a}{r} e_{sx2}^{(0)},\\
&\epsilon_{xs3} = e_{xs3}^{(0)} + \frac{a}{r} e_{sx3}^{(0)},\\
&\epsilon_{xz1} = e_{xz1}^{(0)}, \qquad \epsilon_{xz2} = 0, \qquad \epsilon_{xz3} = 0,\\
&\epsilon_{sz1} = e_{sz1}^{(0)}, \qquad \epsilon_{sz2} = 0, \qquad \epsilon_{sz3} = 0.
\end{aligned}
\tag{12}
$$

In eqs. (11) and (12) $u_x^{(0)}, u_x^{(1)}, \ldots$ are the shell-displacements, $e_{xx}^{(0)}, e_{xx}^{(1)}, \ldots$ the shell-strains, and these are all functions of x and s only. The shell-stresses needed are defined as follows:

$$
\begin{aligned}
&(\sigma_{xxl}^{(0)}, \sigma_{xxl}^{(1)}) = \int_{a_l}^{b_l} \sigma_{xxl}\left(1 + \frac{z}{a}\right)(1, z)\,\mathrm{d}z,\\
&(\sigma_{ssl}^{(0)}, \sigma_{ssl}^{(1)}) = \int_{a_l}^{b_l} \sigma_{ssl}(1, z)\,\mathrm{d}z,\\
&(\sigma_{zzl}^{(0)}, \sigma_{zzl}^{(1)}) = \int_{a_l}^{b_l} \sigma_{zzl}\left(1 + \frac{z}{a}\right)(1, z)\,\mathrm{d}z,\\
&(\sigma_{xsl}^{(0)}, \sigma_{xsl}^{(1)}) = \int_{a_l}^{b_l} \sigma_{xsl}\left(1 + \frac{z}{a}\right)(1, z)\,\mathrm{d}z,
\end{aligned}
\tag{13}
$$

$$(\sigma^{(0)}_{sxl}, \sigma^{(1)}_{sxl}) = \int_{a_l}^{b_l} \sigma_{xsl}(1, z)\, \mathrm{d}z,$$

$$\sigma^{(0)}_{xz1} = \int_{-h_1}^{h_1} \sigma_{xz1}\left(1 + \frac{z}{a}\right) \mathrm{d}z,$$

$$\sigma^{(0)}_{sz1} = \int_{-h_1}^{h_1} \sigma_{sz1}\, \mathrm{d}z,$$

which are also independent of z and in which the limits of integration cover the thickness of the layer.

To derive the equations of the sandwich cylindrical shell we substitute eqs. (10) or (11) and (12) into eq. (9), carry out the integration with respect to z, and make use of eqs. (13) together with the assumption

$$h^2/a^2 \ll 1$$

for a thin shell. Since the variations of the shell-displacements, strains, and stresses are independent and arbitrary, their coefficients in the various integrals must vanish, and the complete system of shell equations is obtained. Thus, the stress equations of motion are

$$\frac{\partial}{\partial x}(\sigma^{(0)}_{xx1} + \sigma^{(0)}_{xx2} + \sigma^{(0)}_{xx3}) + \frac{\partial}{\partial s}(\sigma^{(0)}_{sx1} + \sigma^{(0)}_{sx2} + \sigma^{(0)}_{sx3})$$
$$+ \bar{p}^+_x\left(1 + \frac{h}{a}\right) + \bar{p}^-_x\left(1 - \frac{h}{a}\right) - 2(\varrho_1 h_1 + \varrho_2 h_2)\ddot{u} - \varrho_1 \frac{2h_1^3}{3a}\ddot{\psi} = 0,$$

$$\frac{\partial}{\partial x}(\sigma^{(0)}_{xs1} + \sigma^{(0)}_{xs2} + \sigma^{(0)}_{xs3}) + \frac{\partial}{\partial s}(\sigma^{(0)}_{ss1} + \sigma^{(0)}_{ss2} + \sigma^{(0)}_{ss3}) + \frac{1}{a}(\sigma^{(0)}_{sz1} + \sigma^{(0)}_{sz2} + \sigma^{(0)}_{sz3})$$
$$+ \bar{p}^+_s\left(1 + \frac{h}{a}\right) + \bar{p}^-_x\left(1 - \frac{h}{a}\right) - 2(\varrho_1 h_1 + \varrho_2 h_2)\ddot{v} - \varrho_1 \frac{2h_1^3}{3a}\ddot{\varphi} = 0,$$

$$\frac{\partial}{\partial x}(\sigma^{(0)}_{xz1} + \sigma^{(0)}_{xz2} + \sigma^{(0)}_{xz3}) + \frac{\partial}{\partial s}(\sigma^{(0)}_{sz1} + \sigma^{(0)}_{sz2} + {}^{(0)}_{sz3}) - \frac{1}{a}(\sigma^{(0)}_{ss1} + \sigma^{(0)}_{ss2} + \sigma^{(0)}_{ss3})$$
$$+ \bar{p}^+_z\left(1 + \frac{h}{a}\right) + \bar{p}^-_z\left(1 - \frac{h}{a}\right) - 2(\varrho_1 h_1 + \varrho_2 h_2)\ddot{w} = 0, \qquad (14)$$

$$\frac{\partial}{\partial x}[\sigma^{(1)}_{xx1} + h_1(\sigma^{(0)}_{xx3} - \sigma^{(0)}_{xx2})] + \frac{\partial}{\partial s}[\sigma^{(1)}_{sx1} + h_1(\sigma^{(0)}_{sx3} - \sigma^{(0)}_{sx2})] - \sigma^{(0)}_{xz1}$$
$$+ \bar{p}^+_x h_1\left(1 + \frac{h}{a}\right) - \bar{p}^-_x h_1\left(1 - \frac{h}{a}\right) - \varrho_1 \frac{2h_1^3}{3a}\ddot{u} - \left(\varrho_1 \frac{2h_1^3}{3} + 2\varrho_2 h_1^2 h_2\right)\ddot{\psi} = 0,$$

$$\frac{\partial}{\partial x}[\sigma_{xs1}^{(1)}+h_1(\sigma_{xs3}^{(0)}-\sigma_{xs2}^{(0)})]+\frac{\partial}{\partial s}[\sigma_{ss1}^{(1)}+h_1(\sigma_{ss3}^{(0)}-\sigma_{ss2}^{(2)})]$$

$$+\sigma_{sz1}^{(0)}+\frac{h_1}{a}(\sigma_{sz3}^{(0)}-\sigma_{sz2}^{(0)})+\bar{p}_s^+h_1\left(1+\frac{h}{a}\right)-\bar{p}_s^-h_1\left(1-\frac{h}{a}\right)-\varrho_1\frac{2h_1^3}{3a}\ddot{v}$$

$$-\left(\varrho_1\frac{2h_1^3}{3}+2\varrho_2h_1^2h_2\right)\ddot{\varphi}=0,$$

where the prescribed surface tractions $\bar{p}_x^+$, $\bar{p}_s^+$, $\bar{p}_z^+$ are those at the outer boundary $z=h$, and $\bar{p}_x^-$, $\bar{p}_s^-$, $\bar{p}_z^-$ those at the inner boundary $z=-h$. The boundary conditions at an edge $x=$ const. are

$$\sigma_{xx1}^{(0)}+\sigma_{xx2}^{(0)}+\sigma_{xx3}^{(0)}=\int_{-h}^{h}\bar{p}_x\left(1+\frac{z}{a}\right)dz \qquad \text{or} \quad u=\bar{u},$$

$$\sigma_{xs1}^{(0)}+\sigma_{xs2}^{(0)}+\sigma_{xs3}^{(0)}=\int_{-h}^{h}\bar{p}_s\left(1+\frac{z}{a}\right)dz \qquad \text{or} \quad v=\bar{v},$$

$$\sigma_{xz1}^{(0)}+\sigma_{xz2}^{(0)}+\sigma_{xz3}^{(0)}=\int_{-h}^{h}\bar{p}_z\left(1+\frac{z}{a}\right)dz \qquad \text{or} \quad w=\bar{w},$$

$$\sigma_{xx1}^{(1)}+h_1(\sigma_{xx3}^{(0)}-\sigma_{xx2}^{(0)})=\int_{-h_1}^{h_1}\bar{p}_{x1}\left(1+\frac{z}{a}\right)z\,dz \tag{15}$$

$$+h_1\int_{h_1}^{h}\bar{p}_{x3}\left(1+\frac{z}{a}\right)dz-h_1\int_{-h}^{-h_1}\bar{p}_{x2}\left(1+\frac{z}{a}\right)dz \text{ or } \psi=\bar{\psi},$$

$$\sigma_{xs1}^{(1)}+h_1(\sigma_{xs3}^{(0)}-\sigma_{xs2}^{(0)})=\int_{-h_1}^{h_1}\bar{p}_{s1}\left(1+\frac{z}{a}\right)z\,dz$$

$$+h_1\int_{h_1}^{h}\bar{p}_{s3}\left(1+\frac{z}{a}\right)dz-h_1\int_{-h}^{-h_1}\bar{p}_{s2}\left(1+\frac{z}{a}\right)dz \text{ or } \varphi=\bar{\varphi},$$

and those at an edge $s=$ const. (for an open shell) are

$$\sigma_{sx1}^{(0)}+\sigma_{sx2}^{(0)}+\sigma_{sx3}^{(0)}=\int_{-h}^{h}\bar{p}_x\,dz \qquad \text{or} \quad u=\bar{u},$$

$$\sigma_{ss1}^{(0)}+\sigma_{ss2}^{(0)}+\sigma_{ss3}^{(0)}=\int_{-h}^{h}\bar{p}_s\,dz \qquad \text{or} \quad v=\bar{v},$$

$$\sigma_{sz1}^{(0)}+\sigma_{sz2}^{(0)}+\sigma_{sz3}^{(0)}=\int_{-h}^{h}\bar{p}_z\,dz \qquad \text{or} \quad w=\bar{w},$$

$$\sigma_{sx1} + h_1(\sigma^{(0)}_{sx3} - \sigma^{(0)}_{sx2}) = \int_{-h_1}^{h_1} \bar{p}_{x1}\,dz + h_1 \int_{h_1}^{h} \bar{p}_{x3}\,dz - h_1 \int_{-h}^{-h_1} \bar{p}_{x2}\,dz \quad \text{or} \quad \psi = \bar{\psi},$$

$$\sigma^{(h)}_{ss1} + h_1(\sigma^{(0)}_{ss3} - \sigma^{(0)}_{ss2}) = \int_{-h_1}^{h_1} \bar{p}_{s1}\,dz + h_1 \int_{h_1}^{h} \bar{p}_{s3}\,dz - h_1 \int_{-h}^{-h} \bar{p}_{s2}\,dz \quad \text{or} \quad \varphi = \bar{\varphi}. \tag{16}$$

Likewise, from the generalized variational equation of motion, the relations between the shell-stresses and shell-strains are found to be

$$\sigma^{(0)}_{xx1} = \frac{2E_1 h_1}{1-\nu_1^2}\left(c^{(0)}_{xx1} + \nu_1 c^{(0)}_{ss1} + \frac{h_1^2}{3a} c^{(1)}_{xx1}\right),$$

$$\sigma^{(0)}_{xx2} = \frac{E_2 h_2}{1-\nu_2^2}\left[\left(1 - \frac{h+h_1}{2a}\right) c^{(0)}_{xx2} + \nu_2 c^{(0)}_{ss2}\right],$$

$$\sigma^{(0)}_{ss1} = \frac{2E_1 h_1}{1-\nu_1^2}\left(c^{(0)}_{ss1} + \nu_1 c^{(0)}_{xx1} - \frac{h_1^2}{3a} c^{(1)}_{ss1}\right),$$

$$\sigma^{(0)}_{xx3} = \frac{E_2 h_2}{1-\nu_2^2}\left[\left(1 + \frac{h+h_1}{2a}\right) c^{(0)}_{xx3} + \nu_2 c^{(0)}_{ss3}\right],$$

$$\sigma^{(1)}_{xx1} = \frac{E_1}{1-\nu_1^2}\frac{2h_1^3}{3}\left(c^{(1)}_{xx1} + \nu_1 c^{(1)}_{ss1} + \frac{1}{a} c^{(0)}_{xx1}\right),$$

$$\sigma^{(0)}_{ss2} = \frac{E_2 h_2}{1-\nu_2^2}\left[\left(1 + \frac{h+h_1}{2a}\right) c^{(0)}_{ss2} + \nu_2 c^{(0)}_{xx2}\right],$$

$$\sigma^{(1)}_{ss1} = \frac{E_1}{1-\nu_1^2}\frac{2h_1^3}{3}\left(c^{(1)}_{ss1} + \nu_1 c^{(1)}_{xx1} - \frac{1}{a} c^{(0)}_{ss1}\right), \tag{17}$$

$$\sigma^{(0)}_{ss3} = \frac{E_2 h_2}{1-\nu_2^2}\left[\left(1 - \frac{h+h_1}{2a}\right) c^{(0)}_{ss3} + \nu_2 c^{(0)}_{xx3}\right].$$

$$\sigma^{(0)}_{xs1} = 2\mu_1 h_1\left(c^{(0)}_{xs1} + c^{(0)}_{sx1} + \frac{h_1^2}{3a} c^{(1)}_{xs1}\right),$$

$$\sigma^{(0)}_{xs2} = \mu_2 h_2\left[\left(1 - \frac{h+h_1}{2a}\right) c^{(0)}_{xs2} + c^{(0)}_{sx2}\right].$$

$$\sigma^{(0)}_{sx1} = 2\mu_1 h_1\left((c^{(0)}_{xs1} + c^{(0)}_{sx1} - \frac{h_1^2}{3a} c^{(1)}_{sx1}\right),\quad \sigma^{(0)}_{xs3} = \mu_2 h_2\left[1 + \frac{h+h_1}{2a}\right) c^{(0)}_{xs3} + c^{(0)}_{sx3}\right],$$

$$\sigma^{(1)}_{xs1} = \mu_1 \frac{2h_1^3}{3}\left(c^{(1)}_{xs1} + c^{(1)}_{sx1} + \frac{1}{a} c^{(0)}_{xs1}\right),\quad \sigma^{(0)}_{sx2} = \mu_2 h_2\left[\left(1 + \frac{h+h_1}{2a}\right) c^{(0)}_{sx2} + c^{(0)}_{xs2}\right],$$

$$\sigma^{(1)}_{sx1} = \mu_1 \frac{2h_1^3}{3}\left(c^{(1)}_{xs1} + c^{(1)}_{sx1} - \frac{1}{a} c^{(0)}_{sx1}\right),\quad \sigma^{(0)}_{sx3} = \mu_2 h_2\left[\left(1 - \frac{h+h_1}{2a}\right) c^{(0)}_{sx3} + c^{(0)}_{xs3}\right],$$

$$\sigma_{xz1}^{(0)} = 2\varkappa_1\mu_1 h_1 \epsilon_{xz1}^{(0)},$$
$$\sigma_{sz1}^{(0)} = 2\varkappa_1\mu_1 h_1 \epsilon_{sz1}^{(0)}, \tag{17}$$

where $\varkappa_1$ is a coefficient introduced for the purpose of adjusting the simple thickness-shear frequency of the shell to its value given by the exact theory of elasticity. For an ordinary sandwich plate or cylindrical shell having a weak core, the value of $\varkappa_1$ may be taken equal to 1[3, 7]. For a single-layered plate it was shown by MINDLIN [11] that the value of this coefficient is $\pi^2/12$.

Finally, the relations between the shell-strains and shell-displacements are further found from the generalized variational equation of motion:

$$\begin{aligned}
\epsilon_{xxi}^{(0)} &= \frac{\partial u_{xi}^{(0)}}{\partial x}, & \epsilon_{xx1}^{(1)} &= \frac{\partial u_{x1}^{(1)}}{\partial x},\\
\epsilon_{ssi}^{(0)} &= \frac{\partial u_{si}^{(0)}}{\partial s} + \frac{u_{zi}^{(0)}}{a}, & \epsilon_{ss1}^{(1)} &= \frac{\partial u_{s1}^{(1)}}{\partial s},\\
\epsilon_{xsi}^{(0)} &= \frac{\partial u_{si}^{(0)}}{\partial x}, & \epsilon_{xs1}^{(1)} &= \frac{\partial u_{s1}^{(1)}}{\partial x}\\
\epsilon_{sxi}^{(0)} &= \frac{\partial u_{xi}}{\partial s}, & \epsilon_{sx1}^{(1)} &= \frac{\partial u_{x1}^{(1)}}{\partial s},\\
\epsilon_{xz1}^{(0)} &= \frac{\partial u_{z1}^{(0)}}{\partial x} + u_{x1}^{(1)}, & \epsilon_{sz1}^{(0)} &= \frac{\partial u_{z1}^{(0)}}{\partial s} + u_{s1}^{(1)} - \frac{u_{s1}^{(0)}}{a}.
\end{aligned} \tag{18}$$

Equations (14) to (18) constitute the complete system of equations for the sandwich cylindrical shell. By taking $a = \infty$ the equations reduce to those for the sandwich plate. By taking $h_2 = 0$ these equations further become those for a homogeneous plate or cylindrical shell including the effect of transverse shear deformation. The stress equations of motion and boundary conditions in eqs. (14) to (16) were obtained before in a more general non-linear form from the ordinary variational equation of motion [1], which, however, could not yield directly and simultaneously the stress-strain-displacement relations.

4. Vibrations of Sandwich Plate and Cylindrical Shell

The vibrations of sandwich plates and cylindrical shells have been investigated intensively in recent years [1–10]. One important conclusion that was reached is that, for a sandwich with a weak core, the transverse shear deformation in the core must not be neglected. This will further be demonstrated below. In addition, the coupling among the extensional, transverse,

and thickness-shear motions in a sandwich cylindrical shell executing axially symmetric vibration will be explored, particularly in relation to the vibrations of a sandwich plate, in which case the extensional motion is uncoupled from the flexural (transverse and thickness-shear) motion. Such an analysis will lead to some new and simple expressions for the frequencies of the cylindrical shell.

Substitution of the stress-strain-displacement relations from eqs. (17) and (18) into the stress equations of motion (14) converts the latter into the displacement equations of motion. For axially symmetric free motions the displacement equations may be shown, after some simplifications, to have the following form:

$$\frac{2E_2h_2}{1-\nu_2^2}\left(u''+\frac{h^2}{a}\psi''+\frac{\nu_2}{a}w'\right)=2(\varrho_1h_1+\varrho_2h_2)\ddot{u},$$

$$2\varkappa_1\mu_1h_1(\psi'+w'')-\frac{2E_2h_2}{1-\nu_2^2}\left(\frac{1}{a^2}w+\frac{\nu_2}{a}u'\right)=2(\varrho_1h_1+\varrho_2h_2)\ddot{w}, \tag{19}$$

$$\frac{2E_2h_2h^2}{1-\nu_2^2}\left(\psi''+\frac{1}{a}u''\right)-2\varkappa_1\mu_1h_1(\psi+w')=\left(\varrho_1\frac{2h_1^3}{3}+2\varrho_2h_1^2h_2\right)\ddot{\psi},$$

where the primes indicate differentiation with respect to x. Among the simplifications that have been introduced in the derivation of eqs. (19) are that $r_2r_h \gg 1$ and E_1 is negligible due to the assumption of a weak core and that the contribution of $\ddot{\psi}$ to translatory motion and that of $\ddot{u}$ to rotatory motion are negligible. Furthermore, the assumption of $h^2/a^2 \ll 1$ for a thin shell has been and will be made use of freely wherever applicable.

For a sandwich cylindrical shell having simply supported edges at $x=0$, l and executing axially symmetric vibration, the following form of the shell-displacements may be employed:

$$u=U\cos\frac{\lambda x}{h}e^{i\omega t},$$

$$w=W\sin\frac{\lambda x}{h}e^{i\omega t}, \tag{20}$$

$$\psi=\Psi\cos\frac{\lambda x}{h}e^{i\omega t},$$

where

$$\lambda=\frac{n\pi h}{l},$$

with $n=1, 2, 3 \ldots$ designating the number of half-waves in the length l of the shell. The values of λ are thus discrete for a given finite shell. Equa-

tions (20) are also applicable to a sandwich cylindrical shell of infinite length, in which case it is only necessary to take $\lambda = 2\pi h/L$, L being the wavelength. The parameter λ then has a continuous variation ranging between 0 for infinitely long waves and ∞ for infinitely short waves.

The frequency equation is obtained in the usual manner by substituting eqs. (20) in (19) and setting the determinant of the coefficients of the amplitudes U, W, Ψ equal to zero. The result is

$$
\begin{aligned}
&\Omega_u\Omega_w\Omega_\psi r_{\varrho h} \\
&-\Omega_u\Omega_w(\varkappa_1 + r_2 r_h\lambda^2) - \Omega_w\Omega_\psi r_{\varrho h} r_2 r_h \lambda^2 - \Omega_u\Omega_\psi r_{\varrho h}\left(\frac{h^2}{a^2} r_2 r_h + \varkappa_1\lambda^2\right) \\
&+ \Omega_u r_2 r_h\left[\frac{h^2}{a^2}(\varkappa_1 + r_2 r_h\lambda^2) + \varkappa_1\lambda^4\right] + \Omega_w r_2 r_h \lambda^2(\varkappa_1 + r_2 r_h\lambda^2) \\
&+ \Omega_\psi r_{\varrho h} r_2 r_h\lambda^2\left(\frac{h^2}{a^2} r_2 r_h + \varkappa_1\lambda^2\right) \\
&- r_2^2 r_h^2\lambda^2\left[\frac{h^2}{a^2}(\varkappa_1 + r_2 r_h\lambda^2)(1 - \nu_2^2) + \varkappa_1\lambda^4\right] = 0,
\end{aligned}
\tag{21}
$$

where

$$\Omega_u = \Omega_w = \Omega_\psi = \frac{\varrho_1}{\mu_1}\omega^2 h^2(1 + r_\varrho r_h),$$

$$r_{\varrho h} = \frac{1/3 + r_\varrho r_h}{(1 + r_h)^2(1 + r_\varrho r_h)},$$

$$r_\varrho = \frac{\varrho_2}{\varrho_1}, \qquad r_h = \frac{h_2}{h_1}, \qquad r_2 = \frac{1}{\mu_1}\frac{E_2}{1 - \nu_2^2}.$$

The subscript u, w or ψ attached to the dimensionless frequency parameter Ω indicates whether the particular Ω is derived from the $\bar{u}$-, $\bar{w}$-, or $\bar{\psi}$-term in eqs. (19).

Equation (21) is cubic in Ω and for a given shell yields three frequencies for any value of λ. When the shell is of infinite length, the equation leads to three real continuous branches of the frequency spectrum for the full range of λ. When the shell is finite and simply supported, these become three families of discrete frequency values, each infinite in number. In general, the extensional, transverse, and thickness-shear motions corresponding to the $\bar{u}$-, $\bar{w}$-, and $\bar{\psi}$-terms are coupled together.

For $\lambda = 0$, the motions become uncoupled, and eq. (21) yields

$$\Omega_{u0} = 0, \quad \Omega_{w0} = \frac{h^2}{a^2} r_2 r_h, \quad \Omega_{\psi 0} = \frac{\varkappa_1}{r_{\varrho h}}. \tag{22}$$

These give the cut-off frequencies of the infinite shell, which are usually the lowest frequencies obtainable from the three branches. For $\lambda \to \infty$ the motions again become uncoupled, and eq. (21) gives

$$\Omega_{u\infty} = r_2 r_k \lambda^2, \quad \Omega_{w\infty} = \varkappa_1 \lambda^2, \quad \Omega_{\psi\infty} = r_2 r_k \lambda^2 / r_{\rho k}. \tag{23}$$

These are the same as those for a sandwich plate, as the wavelength is now extremely small. It is seen that Ω_{u0} is always smaller than Ω_{w0}. The latter in turn is usually smaller than $\Omega_{\psi 0}$, although the reverse can be true. On the other hand, since

$$r_2 r_k \gg \varkappa_1, \quad r_{\rho k} < 1,$$

we always have

$$\Omega_{\psi\infty} > \Omega_{u\infty} > \Omega_{w\infty}.$$

For $h/a = 0$ the sandwich cylindrical shell degenerates into a sandwich plate, and eq. (21) becomes uncoupled into

$$\Omega_u - r_2 r_k \lambda^2 = 0, \tag{24}$$

$$\begin{aligned} &\Omega_w \Omega_\psi r_{\rho k} - \Omega_w(\varkappa_1 + r_2 r_k \lambda^2) \\ &-\Omega_\psi r_{\rho k} \varkappa_1 \lambda^2 + \varkappa_1 r_2 r_k \lambda^4 = 0. \end{aligned} \tag{25}$$

These frequency equations are for extensional and flexural vibrations, respectively, of the plate [4, 9]. The cut-off frequencies for $\lambda = 0$ are now given by

$$\Omega_{u0} = 0, \quad \Omega_{w0} = 0, \quad \Omega_{\psi 0} = \frac{\varkappa_1}{r_{\rho k}} \tag{26}$$

which are also deducible from eqs. (22) by putting $h/a = 0$. The value $\Omega_{u0} = 0$ common to the sandwich plate and cylindrical shell corresponds to translatory motion in the x-direction of the structure as a whole. The value $\Omega_{w0} = (h^2/a^2) r_2 r_k$ for the sandwich cylindrical shell corresponds to a ring-type symmetric motion, which in the case of the plate degenerates into a translatory motion of the plate as a whole in the transverse direction. The frequency Ω_{w0} for the sandwich plate is thus zero, as given by eqs. (26). Because the shell is thin, its simple thickness-shear frequency $\Omega_{\psi 0}$ becomes essentially the same as that of the sandwich plate, as shown by eqs. (22) and (26). For $\lambda \to \infty$ eqs. (24) and (25) for the sandwich plate also yield eqs. (23), which should be good for both the plate and shell, as was pointed out earlier.

For arbitrary values of h/a and λ eq. (21) cannot be uncoupled and the three roots of the frequency must be calculated simultaneously. How-

ever, if ν_3^2 is negligible in comparison with one, which will be assumed to be the case, eq. (21) does become separable into the following two equations:

$$\Omega_u = r_2 r_h \lambda^2, \tag{27}$$

$$\Omega_w \Omega_\psi r_{\varrho h} - \Omega_w(\varkappa_1 + r_2 r_h \lambda^2) - \Omega_\psi r_{\varrho h}\left(\frac{h^2}{a^2} r_2 r_h + \varkappa_1 \lambda^2\right) + r_2 r_h\left[\frac{h^2}{a^2}(\varkappa_1 + r_2 r_h \lambda^2) + \varkappa_1 \lambda^4\right] = 0. \tag{28}$$

Equation (27) for extensional motion of the shell is the same as eq. (24) for the sandwich plate, and eq. (28) for flexural motion is reducible to eq. (25) for the plate on putting $h/a = 0$. The latter is a quadratic equation and may be solved explicitly to give

$$\Omega_w, \Omega_\psi = \frac{1}{2r_{\varrho h}}\left\{\left(\varkappa_1 + \frac{h^2}{a^2} r_2 r_h r_{\varrho h}\right) + (r_2 r_h + \varkappa_1 r_{\varrho h})\lambda^2 \mp \left[\left(\varkappa_1 - \frac{h^2}{a^2} r_2 r_h r_{\varrho h}\right) + (r_2 r_h - \varkappa_1 r_{\varrho h})\lambda^2\right] \times \left[1 + \frac{4\varkappa_1^2 r_{\varrho h} \lambda^2}{\left\{\left(\varkappa_1 - \frac{h^2}{a^2} r_2 r_h r_{\varrho h}\right) + (r_2 r_h - \varkappa_1 r_{\varrho h})\lambda^2\right\}^2}\right]^{1/2}\right\}. \tag{29}$$

Now the fraction in the square root in eq. (29) is always much smaller than one as long as the denominator does not become very small, which is always true in ordinary cases in which

$$\varkappa_1 - \frac{h^2}{a^2} r_2 r_h r_{\varrho h} > 0 \quad \text{or} \quad \Omega_{\psi 0} > \Omega_{w0}.$$

In fact, even when $\Omega_{\psi 0} < \Omega_{w0}$, the fraction in eq. (29) often is still small. With exceptions, therefore, the square root in eq. (29) may be replaced by the first two terms of its binomial expansion. After further simplification according to the assumption $r_2 r_h \gg 1$, the following results are obtained:

$$\Omega_w = \frac{h^2}{a^2} r_2 r_h + \frac{\varkappa_1 r_2 r_h \lambda^4}{\varkappa_1 - \frac{h^2}{a^2} r_2 r_h r_{\varrho h} + r_2 r_h \lambda^2}, \tag{30}$$

$$\Omega_\psi = \frac{1}{r_{\varrho h}}(\varkappa_1 + r_2 r_h \lambda^2). \tag{31}$$

When any two of the three expressions in eqs. (27), (30), and (31) yield approximately the same frequency, the effect of coupling becomes strong,

and these equations become less accurate. Thus, equating Ω_u and Ω_w in them yields

$$\lambda = h/a$$

near which value eqs. (27) and (30) are expected to yield less accurate frequency values. Similarly, eqs. (30) and (31) become less accurate near the value

$$\lambda = \left(\frac{h^2}{a^2} r_{\varrho h} - \frac{\varkappa_1}{r_2 r_h}\right)^{1/2}$$

which is obtained by equating Ω_w and Ω_ψ in these equations, but which is real only if $\Omega_{\psi 0} < \Omega_{w0}$. In contrary to these two cases, Ω_u and Ω_ψ are never equal to each other according to eqs. (27) and (31).

To demonstrate the importance of transverse shear deformation in the core of the sandwich we first reduce eq. (30) to the following result for a sandwich plate:

$$\Omega_w = \frac{r_2 r_h \lambda^4}{1 + r_2 r_h \lambda^2/\varkappa_1}. \tag{32}$$

Now the effect of transverse shear deformation may be suppressed by putting $\varkappa_1 = \infty$, as this will make, according to eqs. (17),

$$\dot{e}^{(0)}_{xz1} = \frac{\sigma^{(0)}_{xz1}}{2\varkappa_1 \mu_1 h_1} = 0.$$

Suppressing the transverse shear effect is thus equivalent to neglecting the term $r_2 r_h \lambda^2/\varkappa_1$ against one in eq. (32). But this is seen to be permissible only when

$$\lambda^2 \ll 1/r_2 r_h.$$

With $n = 1$ and $\lambda = \pi h/l$ for the fundamental frequency, this yields

$$2h/l \ll 2/\pi\sqrt{(r_2 r_h)}.$$

For $r_2 r_h = 100$, which is not uncommon and is in fact often exceeded, the thickness-span ratio of the sandwich plate has to be limited to

$$2h/l \ll 1/15.7.$$

For greater value of $r_2 r_h$ and for higher frequencies ($n = 2, 3, \ldots$) the limitation becomes even more severe. The limitation can be somewhat relaxed in the case of a sandwich cylindrical shell executing symmetric transverse vibration, but the transverse shear effect in general must not be neglected.

To illustrate the use of eqs. (27), (30), and (31) we consider the numerical cases in which

$$r_\varrho = 34.4, \quad r_h = 0.1, \quad r_2 = 4790; \quad \varkappa_1 = 1, \quad \nu_2 = 0.$$
$$2h/a = 0.1/30.$$

While $2h/a = 0$ refers to the sandwich plate, $2h/a = 1/30$ is for a sandwich cylindrical shell. The results of eqs. (27), (30), and (31) are shown in Fig. 1, where λ is given a continuous variation, although the results are appli-

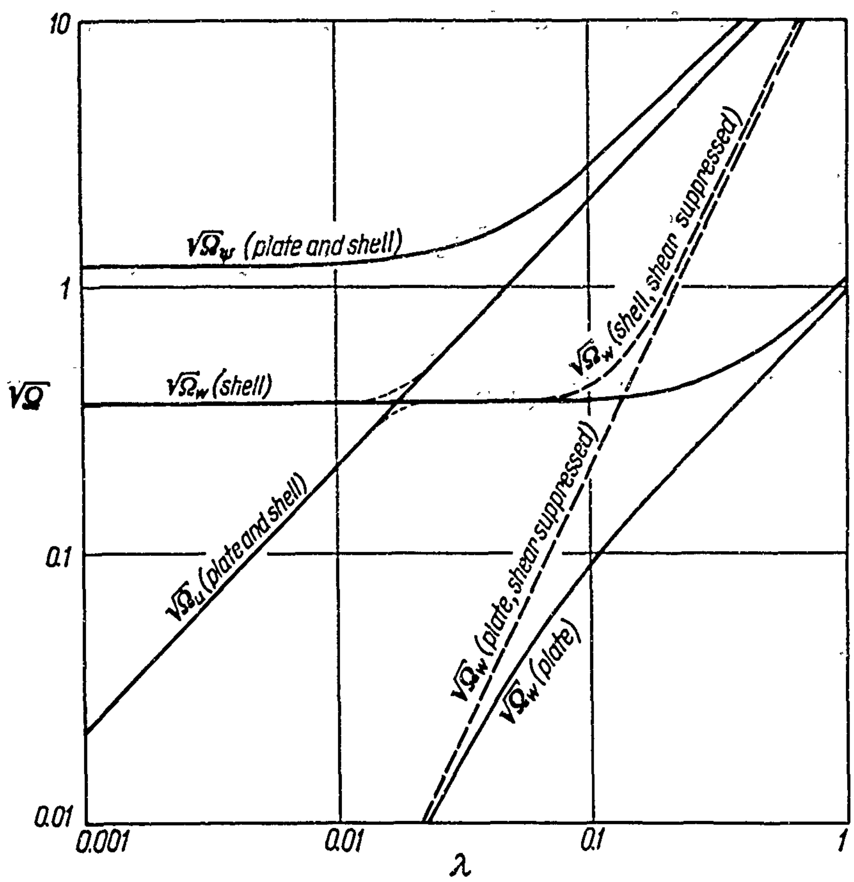

Fig. 1. Frequency parameters.

cable to simply supported sandwich plates and cylindrical shells having finite lengths as well as to the infinitely long ones. For the cylindrical shell the $\sqrt{\Omega_u}$-and $\sqrt{\Omega_w}$-branches do not actually cross each other, even when $\nu_2 = 0$. They appear to cross each other only because of the approximate nature of eqs. (27) and (30). Precisely speaking, near the crossing between the

two branches where $\lambda = h/a = 1/60$ the solid curves should be replaced by the dotted ones also shown in the figure, and the $\sqrt{\Omega_w}$-branch (or $\sqrt{\Omega_u}$-branch) to the left of the crossing should be connected to the $\sqrt{\Omega_u}$-branch (or $\sqrt{\Omega_w}$-branch) to the right of the crossing. In other words, the solid curves given by eqs. (27) and (30) actually serve as bounds to the true branches of the frequency spectrum of the infinite shell. For a simply supported finite shell for which λ takes only discrete values, the true picture is not distorted as much, and it is only important to remember that the frequency values given by eqs. (27) and (30) are not so accurate for values of λ near h/a as for other values of λ.

In Figure 1 are also shown the results for $\sqrt{\Omega_w}$ with the effect of transverse shear deformation neglected. For $\lambda = 0.08$ ($n = 1$, $2h/l = 1/20$, for instance) the frequency of transverse vibration is 101 per cent too high for the sandwich plate but only 5 per cent too high for the cylindrical shell when the transverse shear effect is neglected. However, for $\lambda = 0.16$ ($n = 2$ for the same $2h/l = 20$, for instance) the errors increase to 259 and 72 per cent for the plate and shell, respectively.

5. Viscoelastic Damping of Vibrations

For damped vibrations the displacements in eqs. (20) are replaced by

$$u = U \cos\frac{\lambda x}{h} e^{i\omega t} e^{-\alpha t}, \qquad w = \ldots,\ \psi = \ldots \tag{33}$$

The logarithmic decrement δ is then given by

$$\delta = \frac{2\pi\alpha}{\omega}$$

which is a measure of the damping of the vibration of the composite plate or shell. The change from eqs. (20) to (33) involves the replacement of ω by $\omega + i\alpha = \omega(1 + i\delta/2\pi)$. Equivalently, the frequency parameter Ω may be replaced by $\Omega(1 + i\delta/2\pi)^2$. The damping property of a viscoelastic material is specified through the use of the complex modulus. In the present study this involves the complex moduli of the core and face materials

$$\mu_1(1 + ig_1), \qquad \frac{E_2}{1 - \nu_2^2}(1 + ig_2)$$

which replace μ_1 and $E_2/(1 - \nu_2^2)$ in the undamped case and in which g_1 and g_2 are the loss factors of the core material in shear and the face material

in extension, respectively. Equivalently, the parameters $\varkappa_1$ and r_2 in the above results may be replaced by $\varkappa_1(1+ig_1)$ and $r_2(1+ig_2)$.

With replacements made, eqs. (27), (30), and (31) become

$$
\begin{aligned}
&\Omega_u\left(1-\frac{\delta_u^2}{4\pi^2}+i\frac{\delta_u}{\pi}\right)=r_2r_h\lambda^2(1+ig_2),\\
&\Omega_w\left(1-\frac{\delta_w^2}{4\pi^2}+i\frac{\delta_w}{\pi}\right)=\frac{h^2}{a^2}r_2r_h(1+ig_2)\\
&\qquad+\frac{\varkappa_1r_2r_h\lambda^4(1+ig_1)(1+ig_2)}{\varkappa_1(1+ig_1)+r_2r_h\left(\lambda^2-\frac{h^2}{a^2}r_{\varrho h}\right)(1+ig_2)},\\
&\Omega_\psi\left(1-\frac{\delta_\psi^2}{4\pi^2}+i\frac{\delta_\psi}{\pi}\right)=\frac{1}{r_{\varrho h}}[\varkappa_1(1+ig_1)+r_2r_h\lambda^2(1+ig_2)].
\end{aligned}
\tag{34}
$$

It will now be assumed that δ, g_1, and g_2 are small enough so that δ^2, g_1^2, g_2^2, and g_1g_2 are negligible in comparison with one. The real parts of eqs. (34) then show that viscoleastic damping has a negligible effect on the frequencies. On the other hand, the imaginary parts of these equations when divided by the corresponding real parts yield the following results for the logarithmic decrements:

$$
\begin{aligned}
\frac{\delta_u}{\pi}&=g_2,\\
\frac{\delta_w}{\pi}&=\frac{g_1}{\Omega_w}\frac{\varkappa_1r_2^2r_h^2\lambda^4\left(\lambda^2-\frac{h^2}{a^2}r_{\varrho h}\right)}{\left[\varkappa_1+r_2r_h\left(\lambda^2-\frac{h^2}{a^2}r_{\varrho h}\right)\right]^2}\\
&\quad+\frac{g_2}{\Omega_w}\left\{\frac{h^2}{a^2}r_2r_h+\frac{\varkappa_1^2r_2r_h\lambda^4}{\left[\varkappa_1+r_2r_h\left(\lambda^2-\frac{h^2}{a^2}r_{\varrho h}\right)\right]^2}\right\},\\
\frac{\delta_\psi}{\pi}&=\frac{g_1\varkappa_1+g_2r_2r_h\lambda^2}{\varkappa_1+r_2r_h\lambda^2}.
\end{aligned}
$$

Introducing the damping parameters $k_{u1}, k_{u2}, \ldots$ by writing

$$
\frac{\delta_u}{\pi}=k_{u1}g_1+k_{u2}g_2,\qquad \frac{\delta_w}{\pi}=\ldots,\qquad \frac{\delta_\psi}{\pi}=\ldots \tag{35}
$$

we find

$$
k_{u1}=0,\quad k_{u2}=1. \tag{36}
$$

$$k_{w1}=\frac{1}{\Omega_w}\,\frac{\varkappa_1 r_2^2 r_h^2 \lambda^4\left(\lambda^2-\frac{h^2}{a^2}r_{\varrho h}\right)}{\left[\varkappa_1+r_2 r_h\left(\lambda^2-\frac{h^2}{a^2}r_{\varrho h}\right)\right]^2}, \tag{37}$$

$$k_{w2}=\frac{1}{\Omega_w}\left\{\frac{h^2}{a^2}r_2 r_h+\frac{\varkappa_1^2 r_2 r_h \lambda^4}{\left[\varkappa_1+r_2 r_h\left(\lambda^2-\frac{h^2}{a^2}r_{\varrho h}\right)\right]^2}\right\},$$

$$k_{\psi1}=\frac{\varkappa_1}{\varkappa_1+r_2 r_h\lambda^2},\qquad k_{\psi2}=\frac{r_2 r_h\lambda^4}{\varkappa_1+r_2 r_h\lambda^2}; \tag{38}$$

where

$$\Omega_w=\frac{h^2}{a^2}r_2 r_h+\frac{\varkappa_1 r_2 r_h\lambda^4}{\varkappa_1+r_2 r\left(\lambda^2-\frac{h^2}{a^2}r_{\varrho h}\right)}.$$

The concept of the damping parameter is a useful one. Since the product between a damping parameter and the related material loss-factor represents the contribution to the damping of the composite structure, the former is a measure of the effectiveness and relative importance of the associated type of viscoelastic damping. The sum of all damping parameters in each case is equal to one; namely,

$$k_{u1}+k_{u2}=1,\quad \ldots,\quad \ldots \tag{39}$$

Equations (39) together with (35) show that, if the loss factors in a certain case are equal to each other and equal to g, then δ/π in that case is also equal to g. Since the loss factors are generally not the same, their contributions to the damping are weighted according to the ratio between the damping parameters, and δ/π cannot be larger than the largest loss factor. It is also noted that the effectiveness of one type of damping will increase at the expense of a decrease in the effectiveness of the other type. On the other hand, the various types of damping may supplement each other. Only two types of damping have been considered in the present treatment, the shear damping of the core and extensional damping of the face layers. There are occasions in which more than these two types have to be taken into account [8].

To induce damping, a highly viscoelastic material is often used for the core of a sandwich. But shear damping of the core is seen to be totally ineffective for the extensional vibration of a sandwich plate or cylindrical shell, because k_{u1} is identically zero according to eqs. (36). Shear damping is also not very effective for low-frequency transverse vibrations, as eqs.

(37) show that k_{w1} diminishes with λ. On the other hand, it is not difficult to see from eqs. (38) that shear damping is most effective for thickness-shear vibrations when the frequency is near the simple thickness-shear frequency. This is so because the type of motion that is predominant is precisely what is needed for shear damping to develop strongly. Whenever shear damping is not effective, extensional damping of the face layers may be resorted to. If the face material has only a very small loss factor, viscoelastic damping layers may have to be bonded to the faces of the sandwich, but this will change the problem to a radically different one.

Among eqs. (36) to (38) only eqs. (37) depend on the effect of curvature, which makes k_{w1} smaller and k_{w2} greater than the corresponding values for a sandwich plate. For the latter, eqs. (37) reduce to

$$k_{w1} = \frac{r_2 r_h \lambda^2}{\varkappa_1 + r_2 r_h \lambda^2}, \qquad k_{w2} = \frac{\varkappa_1}{\varkappa_1 + r_2 r_h \lambda^2}.$$

Comparison with eqs. (38) reveals that we now have

$$k_{w1} = k_{\psi 2}, \qquad k_{w2} = k_{\psi 1}.$$

These interesting results have not been observed before and hold true only for the sandwich plate but not for the cylindrical shell. They indicate that an increase in the effectiveness of damping of the transverse vibration is always accompanied by an equal decrease in the effectiveness of damping of the thickness-shear vibration, the increase and decrease in damping effectiveness being measured in terms of the damping parameters. A similar conclusion may be drawn for the sandwich cylindrical shell, but the increase and decrease are not generally equal.

To demonstrate our findings on viscoelastic damping, the damping parameters as given by eqs. (37) and (38) have been calculated for the same numerical cases discussed before. The results are shown in Fig. 2, in which $k_{\psi 1}$ or k_{w2} is measured from the bottom line to a curve, and $k_{\psi 2}$ or k_{w1} measured from the top line to the curve. Numerical results of eqs. (36) are trivial and need not be plotted. In addition to having verified the above discussion on damping parameters, the results in Fig. 2 further reveal the fact that shear damping contributed by a viscoelastic core can be quite ineffective for a sandwich cylindrical shell. Such is the case, for instance, when λ is in the neighborhood of 0.14, for which Fig. 2 gives

$$k_{w1} = 0.13, \qquad k_{\psi 1} = 0.10.$$

Neither of these two values is a large number.

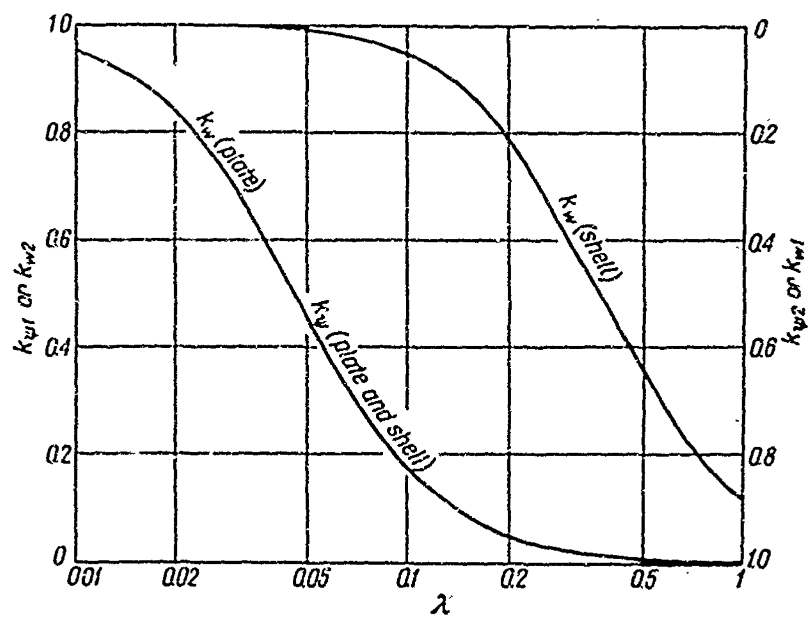

Fig. 2. Damping parameters.

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